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THE PRINCIPLE OF LEAST ACTION AND PERIODIC SOLUTIONS IN PROBLEMS OF CLASSICAL MECHANICS

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The problem of existence of periodic solutions of equations of natural mechanical system motions is considered in the case when region D of all possible motions is bounded. Periodic libration solutions are derived for systems with many degrees of freedom. The trajectory of such solution is diffeomorphic to segment [0, 1], its ends lie at the boundary of D, and the representative point oscillates along that curve. Existence of libration solutions is proved in the case when the region of possible motions is diffeomorphic to the direct product $N \times [0, 1]$, where N is a smooth compact manifold. Obtained results are applied in the problem of motion of a solid body with a fixed point in a Newtonian force field.

1. Statement of the problem. Let M be a smooth compact n-dimensional manifold representing the configuration space of a natural mechanical system with n degrees of freedom. We denote the kinetic energy by T (a smooth function in the tangential stratification of the configuration space and quadratic with respect to velocities) and the potential by V (a smooth function in M). The first integral of motion of the system is that of energy $T - V = h$. For a fixed h we determine from this integral the region $D = \{h + V \geqslant 0\} \subset M$. By the principle of least action in region D the problem of derivation of solutions of equations of motion reduces to that of determination of geodetic lines of the following metric:

$$dp^2 = (h + V)\cdot ds^2$$

where ds^2 is the Riemannian metric in M that specifies the kinetic energy (i.e. $T = {}^1/_2\,(ds/dt)^2$).

Two cases must be considered viz: (1) $h > \max_M(-V)$, and (2) $h \leqslant \max_M(-V)$. In the first case D coincides with the whole configuration space and the problem of existence of periodic solutions of equations of motion reduces to finding closed geodetic lines of the smooth Riemannian manifold (M, dp). To each closed geodetic line correspond two different periodic solutions of the input problem (of motion along such curves in opposite directions). By analogy with systems with a single degree of freedom we call these solutions gyrations. Existing estimates of the number of closed geodetic lines partly depend on the topological structure of M and partly on the Riemannian metric dp [1]. So far the best universal lower estimate is 2 [2]. Thus at least four different periodic solutions exist at $(2n - 1)$-dimensional levels of the energy integral at constant $h > \max(-V)$

In the second case region D is bounded and the metric $d\rho$ has the singularity that the closer one comes to the boundary of D, the shorter becomes the length. The length of any curve lying on the boundary itself $(V = -h)$ is zero.

We denote the boundary of manifold $N \subset M$ by ∂N. Henceforth only such constant energies h are considered for which there are no critical points of potential V along ∂D. Any other values of h, and in particular $h = \max(-V)$ are critical. The metric of the manifold critical values is zero [3]. Equilibrium states of the considered system exist along ∂D for critical h, and singular points of equations of motion are present on the corresponding levels of the energy integral. For noncritical h the boundary ∂D is a smooth compact $(n - 1)$-dimensional manifold.

Let us consider the problem of existence of periodic solutions at the related energy levels. The problem of existence of closed geodetic lines in a bounded Riemannian manifold, which does not have common points with the boundary of that manifold, was considered by Whittaker [4] and by Birkhoff [5]. Since in these investigations the nondegeneracy of the metric at the boundary and the convexity of the boundary itself were specifically stipulated, their results are evidently inapplicable in the problem stated here.

2. Libration in systems with many degrees of freedom. We denote the position of the system in the configuration space M by m $(m \in M)$. Let $q = \{q_i\}$ $(i = 1 \ldots n)$ be some local coordinates in M.

Lemma 1. If $q_1(t)$ and $q_2(t)$ are two solutions of the equation of motion with initial data $q_1(0) = q_2(0) = q_0$ and $q_1^{\cdot}(0) = -q_2^{\cdot}(0) = v_0$, then $q_1(\pm t) = q_2(\mp t)$.

Corollary. If $q(t)$ is the solution of equations of motion with initial conditions $q(0) = q_0$ and $q^{\cdot}(0) = 0$, then $q(t) = q(-t)$.

Proof of Lemma 1. If $q(t)$ is the solution of Lagrange equations with the Lagrangian $L = T + V$ and initial conditions (for $t = 0$) $q(0) = q_0$ and $q^{\cdot}(0) = v_0$, then $q(-t)$ is the solution of the same equations with initial conditions $q(0) = q_0$ and $q^{\cdot}(0) = -v_0$. To complete the proof it is necessary to use the theorem of uniqueness of solutions of Lagrange equations with positive definite quadratic form of T.

Lemma 2. Solution of the equations of motion whose trajectory intersects the boundary ∂D at more than two different points does not exist.

Corollary. If for $t = 0$ point $m \in \partial D$, then there exists $\varepsilon > 0$ such that for $t \in (0, \varepsilon)$ point $m \notin \partial D$.

Proof of Lemma 2. Let us assume that there exists a trajectory that successively intersects ∂D at three points a, b and c. Point m moving from point a reaches after some time point b. Then, in accordance with the corollary to Lemma 1, point m moves along the same trajectory in the opposite direction, and after some time returns to point a, after which point m will move again from point a to point b, and so on. Hence point m can never reach point c. This proves Lemma 2.

Theorem 1. If the trajectory of a certain solution of the equations of motion has two common points with ∂D, there are no other common points and the solution is periodic.

Proof. Let γ be the trajectory of such solution. According to Lemma 2 curve γ has only two common points with ∂D, and point m (by the corollary of Lemma 1) periodically oscillates between the ends of γ.

By analogy with systems with one degree of freedom we call the solutions described in Theorem 1 libration solutions.

3. Construction of the sequence of geodetic line segments.

We shall prove the existence of librations in the case when region D of possible motions is diffeomorphic to the direct product $N \times [0, 1]$, where N is a smooth $(n-1)$-dimensional manifold. The boundary ∂D consists of two manifolds $\partial D'$ and $\partial D''$ diffeomorphic to N. Without loss of generality N can be considered as a connected manifold.

The following method is proposed for the derivation of libration periodic solutions. We consider in a fixed region $D = N \times [0, 1]$ the sequence of mutually imbedded subregions

$$D_k = N \times [1 / (k+2),\ 1 - 1 / (k+2)] \quad (k = 1, 2...)$$

$$D_1 \subset D_2 \subset \ldots \subset D_k \subset \ldots \subset D$$

whose boundaries ($\partial D_k'$ and $\partial D_k''$) for $k \to \infty$ tend uniformly to ∂D. For each region D_k segments of geodetic lines in metric $d\rho$ with their ends on ∂D_k are constructed. It is then shown that it is possible to choose from the sequence of constructed segments a subsequence which in region D is convergent in metric dS to the geodetic line whose ends lie in ∂D, and the motion on that geodetic line is periodic.

Theorem 2. A segment of the geodetic line γ_k in metric $d\rho$ whose ends lie on $\partial D_k'$ and $\partial D_k''$ exists in region D_k $(k = 1, 2 \ldots)$, and the lengths of γ_k are uniformly upper bound with respect to k.

Proof. Let us consider the smooth manifold $M' = N \times R$. We identify the submanifold $N \times [0, 1] \subset M'$ with region D, fix the number k, and denote D_k by E. The metric $d\rho$ is determined in E and in some neighborhood of manifold E in M'. Let $d\rho'$ be a smooth metric in M', such that the Riemannian space $(M', d\rho')$ is complete and $d\rho'$ coincides in E with $d\rho$. Such metric exists according to the statement on smooth continuation of tensor fields (see, e.g., [3]). It is evident that the geodetic lines in the new metric $d\rho'$ coincide in E with the geodetic lines in metric $d\rho$.

Let $m_1 \in \partial E'$ and $m_2 \in \partial E''$. The exact lower bound of the length of piecewise smooth curves beginning at m_1 and ending at m_2 is taken as the length $\rho(m_1, m_2)$ between points m_1 and m_2 The lower bound of distances between any points on $\partial E'$ and $\partial E''$ is taken as the distance ρ_E between $\partial E'$ and $\partial E''$. Since $\rho(m_1, m_2)$ is continuous in $\partial E' \times \partial E''$, and $\partial E'$ and $\partial E''$ are compact, hence there exist on $\partial E'$ and $\partial E''$ points a_1 and a_2 the distance between which is ρ_E. Because the Riemannian space $(M', d\rho')$ is complete, points a_1 and a_2 can be connected by the geodetic line $(\gamma = \gamma_k)$ of length ρ_E [3]. Let us show that γ lies entirely in E. If we assume the opposite, then there must exist a part of γ which connects certain points on $\partial E'$ and $\partial E''$ whose length is shorter than ρ_E. Since the length $\gamma = \gamma_k$ does not exceed that of any piecewise-smooth curve in region D whose ends lie on $\partial D'$ and $\partial D''$, the lengths γ_k are uniformly upper bound with respect to k. Theorem 2 is proved.

4. Proof of existence of periodic libration solutions.

Theorem 3. A periodic libration solution exists in region $D = N \times [0, 1]$, whose trajectory has no self-intersections and whose ends lie on $\partial D'$ and $\partial D''$.

Proof. Let $l_1, \ldots, l_k, \ldots$ be the length of geodetic line segments $\gamma_1, \ldots, \gamma_k, \ldots$ Evidently $0 < l_1 < \ldots < l_k < \ldots$. Let $l = \sup_k l_k$. The length of arc p read from one of the ends can be taken as the parameter of curves γ_k. Obviously, $0 \leqslant p \leqslant l_k$. It is, however, more convenient to use another parameter t which for all k varies from 0 to 1 and satisfies the condition $p = l_k t$. The inequalities

$$\rho(\gamma_k(p_1), \gamma_k(p_2)) \leqslant |p_1 - p_2|, \quad \rho(\gamma_k(t_1), \gamma_k(t_2)) \leqslant l_k |t_1 - t_2| \quad (4.1)$$
$$\rho(\gamma_k(t_1), \gamma_k(t_2)) \leqslant l |t_1 - t_2|$$

where $\rho(m_1, m_2)$ is the distance between points m_1 and $m_2 \in D$ in metric dp, are obvious.

We denote by γ_k^n the parts of geodetic lines γ_k, when $t \in [1/n, 1 - 1/n]$ $(n = 3, 4 \ldots)$. We shall show that at the boundaries $\partial D'$ and $\partial D''$ there exist for any n neighborhoods U_n' and U_n'' such that for any k curves γ_k^n have empty intersections with these. Note that curves γ_k^n are at both ends shorter than γ_k by at least $l_1 / n > 0$. Let us assume that there are no neighborhoods U_n' with such properties. Then along curves γ_k^n there exist points a_k that may be arbitrarily close (in metric dS) to $\partial D'$ and, consequently, to $\partial D_k'$. Let us connect point a_k to $\partial D_k'$ by the short length segment A_k, and consider the piecewise-smooth curve γ_k' consisting of that segment and the section of the geodetic line γ_k between point a_k and $\partial D_k'$. For considerable k its length is obviously shorter than l_k. But this contradicts the principle of construction of curves γ_k.

Thus for a fixed n, curves γ_k^n lie within the compact $D' = D \setminus (U_n' \cup U_n'')$. We shall assume that the sequence of points $a_m \in D'$ converges in metric dp (ds) to point $a \in D'$, if the distance between a_m and a in metric dp (ds) tends to zero for $m \to \infty$. Since metric dp is nondegenerate in D', hence the definitions of convergence in metrics dp and ds are equivalent. The manifold of curves γ_k^n is equicontinuous, as implied by formulas (4.1) which are valid also for γ_k^n. Hence, according to the generalized Arzelà theorem it is possible to select from any infinite submanifold of geodetic γ_k^n a subsequence $(\gamma_k^n)_u$ $(u = 1, 2 \ldots)$ which in metric ds uniformly converges to the continuous curve γ^n: $[1/n, 1-1/n] \to D$ [6]. It is obvious that γ^n is geodetic in metric dp.

Let n now assume the values 3, 4, 5.... It is possible to choose from the sequence γ_k^3 $(k = 1, 2, \ldots)$ the subsequence $(\gamma_k^3)_u$ $(u = 1, 2 \ldots)$ which uniformly converges to the geodetic Γ_3: $[^1/_3, ^2/_3] \to D$, and from the infinite manifold $(\gamma_k^4)_u$ $(u = 1, 2, \ldots)$ it is possible to choose the subsequence $((\gamma_k^4)_u)_v$ $(v = 1, 2, \ldots)$ which uniformly converges to the geodetic Γ_4: $[^1/_4, ^3/_4] \to D$. It is obvious that $\Gamma_3 \subset \Gamma_4$. This process can be continued ad infinitum. As the result we obtain the mutually imbedded geodetics

$$\Gamma_3 \subset \Gamma_4 \subset \ldots \subset \Gamma_n \subset \ldots \quad (\Gamma_n: [1/n, 1 - 1/n] \to D)$$

Let us assume

$$\Gamma = \bigcup_n \Gamma_n; \quad \Gamma: (0, 1) \to D$$

is a geodetic line in metric dp. The lengths of Γ_n are uniformly bound above with respect to n by the number $l > 0$. Consequently the length of the geodetic Γ also does not exceed l. Furthermore, two sequences of points m_k' and m_k'' for $k \to \infty$ indefinitely approach $\partial D'$ and $\partial D''$, respectively.

Let us show that curve Γ has no self-intersections by assuming the contrary, i.e. that the equality $\Gamma(t_1) = \Gamma(t_2)$ is valid for some $t = t_1$ and t_2 $(0 < t_1 < t_2 < 1)$ For any $\varepsilon > 0$ there exists $n = n(\varepsilon)$ such that $[t_1, t_2] \subset [1/n, 1 - 1/n]$ and

$\rho(\gamma_n(t), \Gamma(t)) < \varepsilon$ for $t_1 \leqslant t \leqslant t_2$. Instead of γ_n let us consider the piecewise-smooth curve γ_n' that for $t \in [1/n, t_1] \cup [t_2, 1 - 1/n]$ coincides with γ_n, and for $t \in (t_1, t_2)$ with the shortest geodetic which connects points $\gamma_n(t_1)$ and $\gamma_n(t_2)$. We denote by L and L' the lengths of parts γ_n and γ_n' when $t_1 \leqslant t \leqslant t_2$. It is obvious that $L \geqslant l_1 | t_1 - t_2 |$ and $L' < 2\varepsilon$. For small ε the piecewise-smooth curve γ_n' that lies in D_n and connects $\partial D_n'$ and $\partial D_n''$ is shorter than γ_n. But this contradicts the assumption that γ_n is the shortest of all piecewise-smooth curves connecting the boundaries of D_n.

It remains to show that the closure Γ $(\overline{\Gamma})$ is a geodetic with ends on ∂D, and that the motion of point m on Γ is periodic. Let us consider the solution of equations of motion for the following initial conditions: at the instant of time $t = 0$ point m lies on Γ inside D, that the velocity is directed along Γ, and that the magnitude of the latter is determined by the value of total energy h specified above. We assume for definiteness that for $t > 0$ point m moves toward the boundary $\partial D'$ (i.e. passes through points m_k' that are close to $\partial D'$). The following may occur: either within a finite time point m reaches $\partial D'$ or for all $t > 0$ point $m \notin \partial D'$. In the first case point m, according to the corollary of Lemma 2 after reaching $\partial D'$ moves along the same trajectory in the opposite direction toward $\partial D''$. The same alternative occurs here: either at a certain instant of time $m \in \partial D''$, or point m never reaches the boundary. In the first case by Theorem 1 point m oscillates periodically along $\overline{\Gamma}$, which proves the above assertion. There remains, therefore, to consider the case when m in its motion along Γ never reaches ∂D. Let us show that in that case point m asymptotically approaches ∂D (here and in what follows we consider convergence with respect to metric ds). We denote by U_ε the ε-neighborhood of manifold ∂D in metric ds. If m does not tend to ∂D, then there exists an $\varepsilon_0 > 0$ such that for an arbitrarily great t point m lies outside U_{ε_0}. On the other hand, point m passes through points m_k' and m_k' which can be as close to ∂D as desired. Beginning at some number k these points lie in $U_{\varepsilon_0/2}$. In metric $d\rho$ the distances between the points of manifolds $D \setminus U_{\varepsilon_0}$ and $D \cap U_{\varepsilon_0/2}$ are bounded below by some positive number. Hence the length of Γ is infinite. This is, however, not so.

We shall prove now that for some $\varepsilon > 0$ point m cannot remain for an infinitely long time in region $V_\varepsilon = \{h + V \leqslant \varepsilon\}$. We select ε_1 sufficiently small for the potential V to be free of critical points in V_{ε_1}. Since V_{ε_1} is compact, there exists a finite cover of manifold V_{ε_1} consisting of small regions $W_s \subset M$ $(s = 1, \ldots, N)$, which can be entirely represented in Cartesian coordinates. First, let us estimate $V^\cdot$ in V_{ε_1}. We assume s to be fixed and denote the local coordinates in W_s by $q_1, \ldots, q_n$. By the Cauchy-Buniakowski inequality

$$|V^\cdot| = \left|\sum_{i=1}^{n} \frac{\partial V}{\partial q_i} q_i^\cdot\right| \leqslant |\operatorname{grad} V| \sqrt{v}$$

where v is the velocity of point m. Since T is a positive definite quadratic form, the inequalities $c_{1,s}v^2 \leqslant T \leqslant c_{2,s}v^2$ $(c_{1,s},\ c_{2,s} > 0)$ are valid in region W_s. It follows from the energy integral $T = h + V$ that in V_{ε_1} the kinetic energy $T \leqslant \varepsilon_1$. Hence the inequality $| V^\cdot | \leqslant c_{3,s}$ is satisfied in region $V_{\varepsilon_1} \cap W_s$.

Let us set $C_3 = \max_s c_{3,s}$. Then throughout the manifold V_{ε_1} the inequality $| V^\cdot | \leqslant C_3$ is valid. Let for some $\varepsilon > 0$ and $\varepsilon < \varepsilon_1$ the intersection $V_\varepsilon \cap W_s$ be nonempty. We estimate in that region

$$V^{\cdot\cdot} = \sum_{i=1}^{n} \frac{\partial V}{\partial q_i} q_i^{\cdot\cdot} + \sum_{i,j=1}^{n} \frac{\partial^2 V}{\partial q_i \partial q_j} q_i^{\cdot} q_j^{\cdot}$$

Using the Legendre transformation and the canonical equations

$$p_i = \frac{\partial T}{\partial q_i^{\cdot}}, \quad q_i^{\cdot} = \frac{\partial T}{\partial p_i}, \quad p_i^{\cdot} = -\frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i}$$

we obtain

$$V^{\cdot\cdot} = \sum_{i,j=1}^{n} \frac{\partial V}{\partial q_i} \frac{\partial^2 T}{\partial p_i \partial p_j} \left(-\frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} \right) + \sum_{i,j=1}^{n} \frac{\partial^2 V}{\partial q_i \partial q_j} q_i^{\cdot} q_j^{\cdot} =$$

$$\sum_{i,j=1}^{n} \frac{\partial V}{\partial q_i} \frac{\partial^2 T}{\partial p_i \partial p_j} \frac{\partial V}{\partial q_j} + \Phi_2$$

where Φ_2 is a quadratic form with respect to $q_i^{\cdot}$ with restricted coefficients in $V_{\varepsilon_1} \cap W_s$. Hence the inequality $|\Phi_2| \leqslant c_{4,s}'\varepsilon$ $(c_{4,s} > 0)$ holds in region $V_\varepsilon \cap W_s$. The expression

$$\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial V}{\partial q_i} \frac{\partial^2 T}{\partial p_i \partial p_j} \frac{\partial V}{\partial q_j} \tag{4.2}$$

is the scalar square of vector grad V in metric T.

Since T is a positive definite form and function V has no critical points in region V_{ε_1}, there exists a $c_{5,s} > 0$ such that in $V_{\varepsilon_1} \cap W_s$ the sum (4.2) is not less than $c_{5,s} / 2$. Hence the inequality $V^{\cdot\cdot} \geqslant c_{5,s} - c_{4,s}\ \varepsilon$ is valid in region $V_\varepsilon \cap W_s$. Setting $C_4 = \max_s c_{4,s}$, $C_5 = \min_s c_{5,s}$ $(C_4, C_5 > 0)$ we find that the estimate $V^{\cdot\cdot} \geqslant C_5 - C_4\varepsilon$ holds throughout region V_ε. Since C_3, C_4 and C_5 are independent of ε, there exists $\varepsilon_0 > 0$ and $\varepsilon_0 < \varepsilon_1$ such that in V_{ε_0} we have simultaneously

$$|V^{\cdot}| \leqslant C_3, \quad V^{\cdot\cdot} \geqslant C_6 \quad (C_3, C_6 > 0)$$

If at $t = 0$ point m is in region V_{ε_0}, then $h + V \geqslant C_6 t^2 / 2 - C_3 t$, and, consequently, the time during which m remains in the manifold $\{h + V \leqslant \varepsilon_0\}$ does not exceed the positive root of the following equation:

$$C_6 x^2 / 2 - C_3 x = \varepsilon_0$$

The assertion stated above is thus proved. As a corollary we obtain that m cannot asymptotically tend to ∂D when $t \to \infty$. This shows that the second alternative is not possible. Theorem 3 is proved.

5. Application to the problem of rotation of a solid body with a fixed point in a Newtonian force field. This natural mechanical system has three degrees of freedom. Its configuration space is represented by the group $SO(3)$. The problem is invariant under the action of the group of gyrations $g^s (s \in [0, 2\pi))$ about the vertical axis. A cyclic integral — area integral — corresponds to group g^s, its constant is denoted by j.

Let us, first, consider the question of existence of periodic motion of a body in a three-dimensional space. Let $h = \omega$ be the maximum critical value of the energy integral. For $h > \omega$ the region of possible motions coincides with the whole $SO(3)$. At least three different closed geodetic lines exist in any Riemannian $SO(3)$ [1]. Six different periodic motions of the solid body correspond to these. For remaining noncritical h each connected component of the region of possible motions is, according to [7, 8], differomorphic to $T^2 \times [0, 1]$ (T^2 is a two-dimensional torus) or to $S^1 \times D^2$ (S^1 is a circle and D^2 a two-dimensional disk). In the first case, by Theorem 3, there exists at least one periodic libration motion of the body. This periodic solution of equations of motion belongs to the area integral zero level, since for $j \neq 0$ the velocity of the body is never zero. If $\gamma(t)$ is a libration solution, then $g^s(\gamma)$ $(s \in [0, 2\pi))$ is also a periodic libration solution. Since γ is not a permanent gyration, hence for $s \in (0, 2\pi)$ we have $g^s(\gamma) \neq \gamma$. Consequently, a single-parameter set of libration motions exists in region $T^2 \times [0, 1]$.

Let us consider in detail the case of $j = 0$. The presence of the symmetry group makes it possible to reduce the problem to that of a system with two degrees of freedom by factorization with respect to g^s. It is obvious that $SO(3) / g^s = S^2$ (Poisson's sphere). Reducing by Routh's method the order of the system in local generalized coordinates ϑ, φ and ψ (Euler's angles), we obtain a natural system with two degrees of freedom in which

$$T = \frac{a\dot{\vartheta}^2}{2} + b\dot{\vartheta}\dot{\varphi} + \frac{c\dot{\varphi}^2}{2}$$

where

$$\begin{aligned} Ka &= AB \sin^2\vartheta + C\cos^2\vartheta\,(A\cos^2\varphi + B\sin^2\varphi) \\ Kb &= (B - A)\,C\sin\vartheta\cos\vartheta\sin\varphi\cos\varphi \\ Kc &= C\sin^2\vartheta\,(A\sin^2\varphi + B\cos^2\varphi) \\ K &= A\sin^2\vartheta\sin^2\varphi + B\sin^2\vartheta\cos^2\varphi + C\cos^2\vartheta \end{aligned}$$

and V is the Newtonian force field potential.

It can be readily shown that T is a positive definite quadratic form. We shall show that T and V, which are definite for $\vartheta \neq 0, \pi$, are analytically continued over the whole Poisson's sphere. This is obvious for the potential V. Let us consider form T in local coordinates $x = \sin\vartheta\sin\varphi$ and $y = \sin\vartheta\cos\varphi$ on S^2, which do not have singularities at the poles

$$T = \frac{\xi\dot{x}^2}{2} + \eta\dot{x}\dot{y} + \frac{\zeta\dot{y}^2}{2}$$

$$K\xi = \frac{ABx^2}{1 - x^2 - y^2} + BC, \quad K\eta = \frac{ABxy}{1 - x^2 - y^2}, \quad K\zeta = \frac{ABy^2}{1 - x^2 - y^2} + AC$$

$$K = Ax^2 + By^2 + C(1 - x^2 - y^2)$$

Since form T analytically depends on x and y when their values are small, the stated assertion is proved.

The derived here results can be applied to the obtained natural system. For $h > \omega$ the region of possible motions coincides with the complete Poisson's sphere. Since on a two-dimensional Riemannian sphere there are at least three different closed non-self-intersecting geodetic lines, the equations of the reduced system have six different periodic solutions [9]. For the remaining noncritical values of h every connected component of the region of possible motions is either a ring $S^1 \times [0, 1]$ or a disk D^2 [7, 8]. In the first case Theorem 3 shows that there exists at least one libration solution with a non-self-intersecting trajectory. The question of existence of periodic solutions in the second case remains open.

Note. The existence of libration solution in the ring region of the reduced system is obviously the result of libration motions of the body in region $T^2 \times [0, 1] \subset SO(3)$.

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ON THE STABILITY OF PERMANENT ROTATION OF A HEAVY SOLID BODY ABOUT A FIXED POINT

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The permanent rotation of a heavy solid body about its principal axis of inertia with a fixed point is considered. Stability is investigated with the use of the theorem on the stability of Hamiltonian systems with two degrees of freedom in the general elliptic case. It is shown that in the absence of certain resonance relationships in the region of necessary stability conditions, which does not coincide with the region of known sufficient conditions, the first approximation indicates the existence of stability, except possibly, in the case when the parameters of the problem lie on some specific manifolds of the parameter space. Subregions that are free of such exceptional manifolds are indicated in each region of necessary stability conditions.

Necessary stability conditions for permanent rotation about principal axes of inertia of a solid body were investigated by Grammel [3]. Sufficient conditions that matched necessary conditions were obtained by Chetaev in the case of Lagrange integrability [4], and by Rumiantsev in that of Kowalewska integrability [5]. Permanent rotation of a body with arbitrary mass distribution about its principal axis of inertia was considered in [6 — 8], where sufficient stability condi-